only at $r = 0$ excludes terms that depend on the vortex angle, which would prevent a simple treatment of the integral. Because the inflow velocity usually increases with r, for example, Fig. 4 of Ref. 2, Eq. (2) must underestimate U_b. Hovering rotor experiments have shown that the vortex pitch has only two values: $k = k_1$ for $x \le x_1$, where $x_1 = Nk_1/2\pi$, and $k = k_2$ otherwise. This allows rewriting Eq. (2) as

$$U_b = \frac{N\Gamma}{4\pi}\left[\frac{1}{k_1}\int_0^{x_1}\frac{r_v^2}{\left(r_v^2+x^2\right)^{\frac{3}{2}}}\,\mathrm{d}x + \frac{1}{k_2}\int_{x_1}^{\infty}\frac{r_v^2}{k\left(r_v^2+x^2\right)^{\frac{3}{2}}}\,\mathrm{d}x\right] \tag{3}$$

If there were no contraction, the integrals are evaluated easily to give, using Eq. (1),

$$U_b = \frac{U_w}{2}\left[1 + \frac{x_1}{\left(r_b^2+x_1^2\right)^{\frac{3}{2}}}\left\{\frac{k_2}{k_1} - 1\right\}\right] \tag{4}$$

Since $k_2 > k_1$, $k_2/k_1 - 1$ is positive, then $U_b/U_w > \frac{1}{2}$. If $k = k_2$ throughout the wake, then $U_b/U_w = \frac{1}{2}$. However, to the level of approximation of the vortex structure,

$$U_b/U_w = (r_w/r_b)^2 \tag{5}$$

so that the change in the pitch is significant. A simple assessment of the effects of contraction can be made by writing

$$U_b > \frac{U_w}{2}\int_0^{\infty}\frac{r_v^2}{\left(r_v^2+x^2\right)^{\frac{3}{2}}}\,\mathrm{d}x \tag{6}$$

using Eq. (1) and $k_2 > k_1$. If the integrand in Eq. (6) is expressed in terms of $\mathrm{d}[x(r_v^2+x^2)^{-1/2}]/\mathrm{d}x$, then integration by parts gives

$$U_b > (U_w/2)(1-\delta) \tag{7a}$$

where

$$\delta = -\frac{1}{2}\int_0^{\infty}\frac{x\,\mathrm{d}r_v^2/\mathrm{d}x}{\left(r_v^2+x^2\right)^{\frac{3}{2}}}\,\mathrm{d}x \tag{7b}$$

which must always be positive for a monotonically contracting wake. Result (7) suggests that wake contraction acts to reduce the difference between U_b/U_w and $\frac{1}{2}$.

Numerical Results

Despite its simplicity, it appears impossible to integrate Eq. (2) analytically when it is combined with the empirical equations for r_v, such as Eq. (2) in Ref. 2 and the pitch data from their Table 2. The integral was evaluated numerically, using this Eq. (1) to remove the dependence on Γ. The result was $U_b/\Omega r_b = 0.0424$. Using Eq. (3), this gives $r_w/r_b = (0.0424/0.0786)^{1/2} = 0.74$, which is approximately halfway between the momentum theory value and the experimental result, but is closer to the latter if it is remembered that the analysis underestimates U_b. From Eq. (3), which gives the effect of the pitch change without contraction, $r_w/r_b = 0.87$.

Conclusion

It is concluded that the change in pitch of the tip vortex wake contraction is responsible for the average velocity through a hovering rotor being greater than the value from simple momentum theory. The effect of wake contraction is to reduce this discrepancy.

Acknowledgment

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Divergence and the *p–k* Flutter Equation

Louw H. van Zyl* and Motodi S. Maserumule†
Defencetek, CSIR, Pretoria 0001, South Africa

Nomenclature

A	=	pitch amplitude
a	=	distance from midchord to pitch axis, positive aft, normalized by semichord
b	=	semichord
$C_{M\alpha}$	=	pitching moment coefficient slope, $M_\alpha/(1/2\rho U^2 b^2)$
I	=	mass moment of inertia about pitch axis per unit span
K	=	torsion spring stiffness
k	=	reduced frequency, $\omega b/U$, in general or $\mathrm{Im}(p)/V$ in the example
M	=	aerodynamic moment about pitch axis per unit span
m	=	mass per unit span
p	=	differential operator, $(2b/U)(\mathrm{d}/\mathrm{d}t)$, in general or $(1/\omega_\alpha)(\mathrm{d}/\mathrm{d}t)$ in the example
r_α	=	radius of gyration about the pitch axis, normalized by semichord
t	=	time
U	=	true airspeed
V	=	reduced velocity, $U/(\omega_\alpha b)$
$\alpha(t)$	=	instantaneous pitch angle
μ	=	mass ratio, $m/(\pi\rho b^2)$
ρ	=	air density
ω_α	=	pitch frequency in a vacuum

Introduction

In principle the flutter equation should be able to predict divergence. In Ref. 1 the use of the k method, as implemented in the FAST flutter analysis system, to determine divergence speeds of forward swept wings is described. The p and p–k methods were used in Refs. 2 and 3 to study divergence of restrained and unrestrained systems. In the p–k method, the real part of the eigenvalue p indicates whether the motion grows or dies out, whereas the imaginary part indicates the oscillation frequency. The generalized aerodynamic forces depend on p; therefore, an iterative solution procedure is generally used. In this study, the simplest aeroelastic divergence problem is solved using three different forms of the p–k flutter equation, namely, that of Hassig[4] in which the aerodynamic coefficients depend only on the imaginary part of p; that of Rodden, Harder, and Bellinger[5] which introduces a dependence on the real part of p by dividing the generalized aerodynamic force matrix into an aerodynamic stiffness matrix and an aerodynamic damping matrix; and a form of the p–k flutter equation that is equivalent to the g method of Chen.[6] The latter method employs a first-order

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*Engineer, Defence Aeronautics Programme, P.O. Box 395.
†Scientist, Defence Aeronautics Programme, P.O. Box 395.

expansion from the imaginary axis to approximate the generalized aerodynamic forces for complex values of p. All three forms are solved using a noniterative root search technique similar to that proposed by Chen. The commonly used mode tracking procedure does not have a mechanism to detect additional roots, and the method of successive approximation does not converge for all roots. The positive real roots of the problem using exact aerodynamic coefficients were also found using a root search method.

Example Problem

In Ref. 7 an example is given of a single-degree-of-freedom-system that exhibits divergence. It is of an airfoil free to pitch about an axis located 0.1304 semichords ahead of the midchord ($a = -0.1304$), $\mu = 23.1$ and $r_\alpha = 0.41$. The system has torsional stiffness about the pitch axis but no structural damping. The equation of motion is simply

$$I\ddot{\alpha} + K\alpha = M \tag{1}$$

When it is assumed that the motion of the airfoil is described by

$$\alpha(t) = A\exp(p\omega_\alpha t) \tag{2}$$

the characteristic equation can be normalized to

$$p^2 + 1 = \frac{V^2 C_{M\alpha}(p/V, a)}{2\pi\mu r_\alpha^2} \tag{3}$$

At divergence, $p = 0$, the steady value of $C_{M\alpha}$ may be substituted and the divergence speed solved as 2.29.

Solutions

Closed-form solutions exist for the aerodynamic forces acting on an oscillating airfoil in incompressible flow. In the solutions given by Von Kármán and Sears,[8] the Theodorsen circulation function is expressed in terms of Bessel functions. The two-term approximation of Jones[9] is commonly used instead of the Bessel functions to reduce computational effort. The derivatives of the aerodynamic coefficients at $p = 0$ are infinite for the exact expressions, whereas they are finite for Jones's approximation. Both Rodden, Harder, and Bellinger's form[5] of the p–k flutter equation and the g method of Chen[6] require the derivative of the aerodynamic moment coefficient to be finite at $p = 0$ in order to predict divergence; therefore, the approximation of Jones[9] was used in the present study.

With Hassig's approximation[4] to the aerodynamic forces, Eq. (3) becomes

$$p^2 + 1 = \frac{V^2 C_{M\alpha}(ik, a)}{2\pi\mu r_\alpha^2} \tag{4}$$

The solution is shown in Fig. 1. The frequency of the pitch mode does not go to zero, and divergence is indicated by the appearance of two neutrally stable real roots at the divergence speed. One becomes stable and the other unstable. Also shown in Fig. 1 is the positive real root of Eq. (3) with the exact expression for $C_{M\alpha}(p/V, a)$. The slope of this real root at divergence is finite, in contrast to the infinite slope predicted by Eq. (4). No unstable oscillatory roots of Eq. (3) were found.

In Rodden, Harder, and Bellinger's form[5] of the flutter equation, the aerodynamic moment coefficient is approximated by

$$C_{M\alpha}(p/V, a) \approx \mathrm{Re}[C_{M\alpha}(ik, a)] + (p/V)\{\mathrm{Im}[C_{M\alpha}(ik, a)]/k\} \tag{5}$$

and leads to the equation of motion

$$p^2 - \frac{V}{2\pi\mu r_\alpha^2}\frac{\mathrm{Im}[C_{M\alpha}(ik, a)]}{k}p + 1 - \frac{V^2}{2\pi\mu r_\alpha^2}\mathrm{Re}[C_{M\alpha}(ik, a)] = 0 \tag{6}$$

The solution is shown in Fig. 2. The pitch mode exist up to $V = 2.26$, slightly below the divergence speed. The frequency of the pitch mode decreases smoothly, but does not go to zero. At $V = 1.50$,

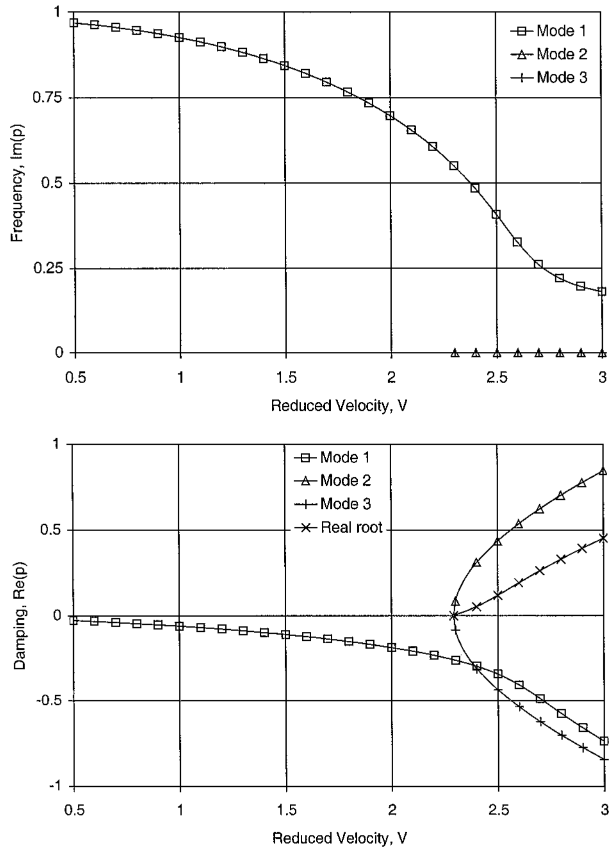

Fig. 1 Solution of Hassig's form[4] of the flutter equation, and exact positive real root.

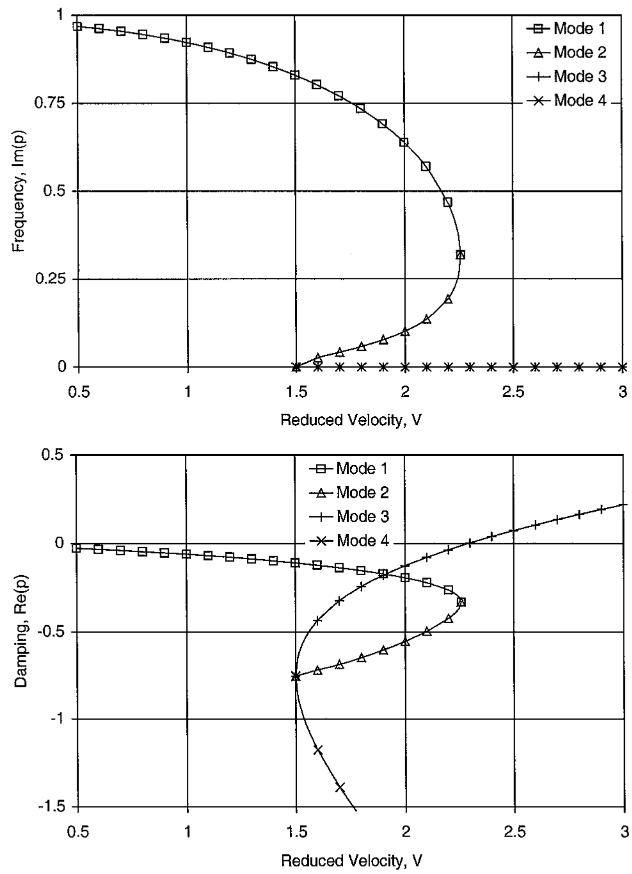

Fig. 2 Solution of Rodden, Harder, and Bellinger's form[5] of the flutter equation.

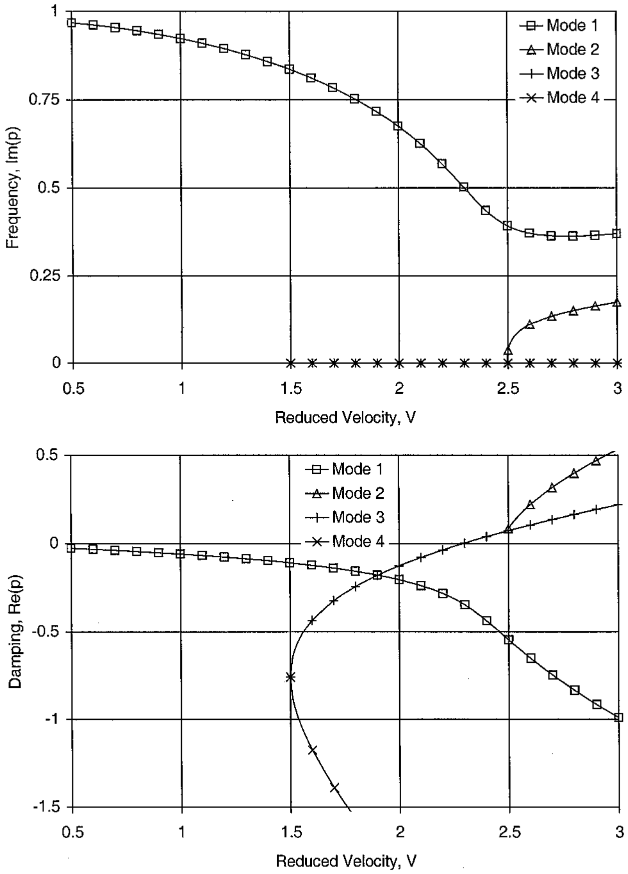

Fig. 3 Solution of the flutter equation with the aerodynamic moment approximated by a first-order expansion from the imaginary axis.

three additional roots appear, one oscillatory and two nonoscillatory. The additional oscillatory mode is stable, and its frequency increases to match that of the oscillatory mode at $V = 2.26$. The two nonoscillatory modes are initially both stable. One becomes more stable and the other becomes less stable, becoming unstable at the divergence speed.

In the g method of Chen[6] the aerodynamic coefficient is approximated by

$$C_{M\alpha}(p/V, a) \approx C_{M\alpha}(ik, a) + (p/V - ik)\frac{\partial C_{M\alpha}(ik, a)}{\partial ik} \tag{7}$$

and leads to the equation of motion

$$p^2 - \frac{V}{2\pi\mu r_\alpha^2}\frac{\partial C_{M\alpha}(ik, a)}{\partial ik}p + 1 - \frac{V^2}{2\pi\mu r_\alpha^2}\left(C_{M\alpha}(ik, a) - ik\frac{\partial C_{M\alpha}(ik, a)}{\partial ik}\right) = 0 \tag{8}$$

The solution is shown in Fig. 3. The pitch mode exists over the entire speed range, and its frequency decreases but does not go to zero. At $V = 1.50$, two stable nonoscillatory roots appear. One becomes more stable and the other becomes less stable, becoming unstable at the divergence speed. At $V = 2.49$, an unstable oscillatory root bifurcates from the divergence root.

Conclusions

The present study shows that divergence is predicted very differently by three forms of the p–k flutter equation even for the simplest of cases. The differences between the results are entirely due to different approximations to the generalized aerodynamic forces. All three forms do, however, predict the same divergence speed. This is to be expected because divergence occurs at $p = 0$, where the three approximations to the pitching moment coefficient are equivalent. However, aeroelastic divergence of free-flying aircraft does not occur at $p = 0$, and we can expect the three different forms to predict different divergence speeds. None of the forms predict that the frequency of the pitch mode goes to zero at divergence, which is in agreement with the analysis in Ref. 7.

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Aeroelastic Divergence and Aerodynamic Lag Roots

Louw H. van Zyl*
Defencetek, CSIR, Pretoria 0001, South Africa

Nomenclature

B	=	damping matrix
c	=	airfoil chord or mean wing chord
g	=	structural damping coefficient, also $2\,\mathrm{Re}(p)c/[\ln(2)V]$
K	=	stiffness matrix
k	=	reduced frequency, $\omega c/2V$
M	=	inertia matrix
m	=	airfoil mass per unit span
p	=	eigenvalue, $\omega(\gamma \pm i)$
Q^I	=	aerodynamic damping matrix, $\mathrm{Im}[Q(k)]$
Q^R	=	aerodynamic stiffness matrix, $\mathrm{Re}[Q(k)]$
$Q(k)$	=	matrix of generalized aerodynamic forces
$\{u\}$	=	vector of degrees of freedom
V	=	true airspeed
γ	=	decay rate coefficient
μ	=	airfoil mass ratio $m/[\pi\rho(c/2)^2]$
ρ	=	air density
ω	=	angular frequency, rad/s

Introduction

FLUTTER analysis methods have been used to predict aeroelastic divergence for a variety of cases, including a hypothetical jet transport wing known as the BAH wing[1–3] and an airfoil with two[4]

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*Engineer, Defence Aeronautics Programme, P.O. Box 395.